Product, Quotient and Chain Rules

Rule 2: Product Rule

Suppose u and v are functions of x.

If $y = uv$,

then $$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}.$$

In words:

First by the derivative of the second + second by the derivative of the first.

Example

If $y = (x^2 - 3x + 2)(x^2 - 2)$, find $\frac{dy}{dx}$.

Solution:

Let $u = x^2 - 3x + 2$ and let $v = x^2 - 2$

$\frac{du}{dx} = 2x - 3$ and $\frac{dv}{dx} = 2x$

$$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx} \quad \text{(product rule)}$$
$$= (x^2 - 3x + 2)(2x) + (x^2 - 2)(2x - 3)$$
$$= 2x^3 - 6x^2 + 4x + 2x^3 - 3x^2 - 4x + 6$$
$$= 4x^3 - 9x^2 + 6$$

Rule 3: Quotient Rule

Suppose u and v are functions of x.

If $y = \frac{u}{v}$

then $$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

In words:

$$\frac{\text{Bottom by the derivative of the top – Top by the derivative of the bottom}}{(\text{Bottom})^2}$$

Example

If $y = \dfrac{x^2}{x-2}$, find $\dfrac{dy}{dx}$.

Solution:

Let $u = x^2$ and let $v = x - 2$

$\dfrac{du}{dx} = 2x$ and $\dfrac{dv}{dx} = 1$

$$\frac{dy}{dx} = \frac{v\dfrac{du}{dx} - u\dfrac{dv}{dx}}{v^2} \qquad \text{(quotient rule)}$$

$$= \frac{(x-2)(2x) - (x^2)(1)}{(x-2)^2}$$

$$= \frac{2x^2 - 4x - x^2}{(x-2)^2}$$

$$= \frac{x^2 - 4x}{(x-2)^2}$$

Note: It is usual practice to simplify the top but **not** the bottom.

Function of a function

When we write, for example, $y = (x+5)^3$, we say that y is a function of x.
If we let $u = (x+5)$, then $y = u^3$, where $u = (x+5)$.
We say that y is a function u, and u is a function of x.
The new variable, u, is the **link** between the two expressions.

Rule 4: Chain Rule

Suppose u is a function of x.

If $y = u^n$

then $\dfrac{dy}{dx} = nu^{n-1}\dfrac{du}{dx}$.

The chain rule should be done in **one** step.

Example

Find $\frac{dy}{dx}$ for each of the following:

(i) $y = (x^2 - 3x)^4$ **(ii)** $y = \frac{3}{2x+5}$

(iii) $y = \sqrt{4x-3}$ **(iv)** $y = \left(x^2 + \frac{1}{x}\right)^3$

Solution:

(i)
$$y = (x^2 - 3x)^4$$
$$\frac{dy}{dx} = 4(x^2 - 3x)^3(2x - 3)$$
$$= (8x - 12)(x^2 - 3x)^3$$

(ii)
$$y = \frac{3}{2x+5}$$
$$y = 3(2x+5)^{-1}$$
$$\frac{dy}{dx} = -3(2x+5)^{-2}(2)$$
$$= \frac{-6}{(2x+5)^2}$$

(iii)
$$y = \sqrt{4x-3}$$
$$y = (4x-3)^{1/2}$$
$$\frac{dy}{dx} = \tfrac{1}{2}(4x-3)^{-1/2}(4)$$
$$= \frac{2}{(4x-3)^{1/2}} = \frac{2}{\sqrt{4x-3}}$$

(iv)
$$y = \left(x^2 + \frac{1}{x}\right)^3$$
$$y = (x^2 + x^{-1})^3$$
$$\frac{dy}{dx} = 3(x^2 + x^{-1})^2(2x - x^{-2})$$
$$= 3\left(x^2 + \frac{1}{x}\right)^2\left(2x - \frac{1}{x^2}\right)$$

Often we have to deal with a combination of the product, quotient or chain rules.

Example

Find $\frac{dy}{dx}$ if **(i)** $y = x\sqrt{9-x^2}$ **(ii)** $y = \sqrt{\frac{1-x}{1+x}}$

Solution:

(i)
$$y = x\sqrt{9-x^2}$$
$$y = x(9-x^2)^{1/2}$$
$$\frac{dy}{dx} = (x)\,.\,\underbrace{\tfrac{1}{2}(9-x^2)^{-1/2}(-2x)}_{\text{(chain rule here)}} + (9-x^2)^{1/2}(1)$$ (product rule and chain rule)

$$= -x^2(9-x^2)^{-1/2} + (9-x^2)^{1/2}$$
$$= \frac{-x^2}{\sqrt{9-x^2}} + \sqrt{9-x^2}$$

(ii) $y = \sqrt{\dfrac{1-x}{1+x}}$

$$y = \left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1-x}{1+x}\right)^{-1/2}\left[\frac{(1+x)(1)-(1-x)(1)}{(1+x)^2}\right] \qquad \left(\text{chain rule followed by the quotient rule}\right)$$

$$= \frac{1}{2}\left(\frac{1+x}{1-x}\right)^{1/2}\left[\frac{-1-x-1+x}{(1+x)^2}\right] \qquad \left(\left(\frac{a}{b}\right)^{-n} = \left(\frac{b}{a}\right)^{n}\right)$$

$$= \frac{(1+x)^{1/2}}{2(1-x)^{1/2}} \cdot \frac{-2}{(1+x)^2}$$

$$= \frac{-1}{(1-x)^{1/2}(1+x)^{3/2}}$$

Exercise 12.2

In questions 1 to 6, use the product rule to find $\dfrac{dy}{dx}$:

1. $y = (2x+3)(x-4)$ **2.** $y = (x+5)(x^2-3x+2)$ **3.** $y = (3x-4)(x^2-2x+3)$

4. $y = (x+3)(x^2-6x+8)$ **5.** $y = (5x^2-3x)(x^2-5x)$ **6.** $y = (3x^3-2x^2+4)(2x-1)$

In questions 7 to 12, use the quotient rule to find $\dfrac{dy}{dx}$:

7. $y = \dfrac{3x+2}{x+1}$ **8.** $y = \dfrac{2x-1}{x+3}$ **9.** $y = \dfrac{3x-1}{x^2-2}$

10. $y = \dfrac{x^2-1}{x^2+1}$ **11.** $y = \dfrac{1-x}{2x-x^2}$ **12.** $y = \dfrac{x^2-x-6}{x^2+x-6}$

In questions 13–18, use the chain rule to find $\dfrac{dy}{dx}$:

13. $y = (3x+2)^4$ **14.** $y = (x^2+2x)^3$ **15.** $y = (2x^2+1)^5$

16. $y = \sqrt{4x+2}$ **17.** $y = \dfrac{1}{2x-5}$ **18.** $y = \dfrac{1}{\sqrt{2x^2-4x}}$

Find $\dfrac{dy}{dx}$ if:

19. $y = x^2(x+3)^4$ **20.** $y = 3x(x+2)^3$ **21.** $y = 3x^2(2x+3)^2$

22. $y = x^2\sqrt{2x+1}$ **23.** $y = x\sqrt{1+x^2}$ **24.** $y = \sqrt{\dfrac{x+1}{x}}$

25. If $f(x) = \sqrt{\dfrac{x}{x+3}}$, find the value of $f'(1)$. **26.** If $f(x) = \sqrt{\dfrac{x-1}{x+1}}$, find the value of $f'(\frac{5}{4})$.